$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

in spherical coordinates. It turns out that $E/H = \sqrt{\mu/\varepsilon}$, the intrinsic impedance of the medium.

17.9 Boundary Conditions

The boundary conditions to be satisfied by the electric and the magnetic fields at an interface separating two media can be obtained from Maxwell's equations. To derive the boundary condition for $\vec{B}$, we consider a pill-box-shaped surface at the interface between medium 1 and medium 2, as shown in Fig. 17.3. Since $\vec{\nabla} \cdot \vec{B} = 0$, we have by applying the divergence theorem over the volume enclosed by the pill-box-shaped surface

$$\oint_S \vec{B} \cdot \hat{n}\, dS = \int_{S_1} \vec{B} \cdot \hat{n}_1\, dS + \int_{S_2} \vec{B} \cdot \hat{n}_2\, dS + \int_{S_3} \vec{B} \cdot \hat{n}_3\, dS = 0. \tag{17.66}$$

When the height of the pill-box is very small, the side surface area S_3 is negligible, so that for bounded $\vec{B}$, the last term on the right-hand side of Eq. (17.66) vanishes. Since $\hat{n}_1$ and $\hat{n}_2$ are oppositely directed, we immediately get

$$B_{1n} = B_{2n}. \tag{17.67}$$

That is, the *normal component of the magnetic induction is continuous across the interface.*

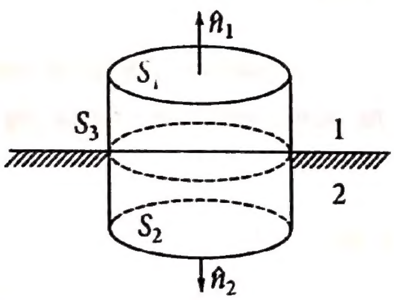

Fig. 17.3. Derivation of boundary conditions on $\vec{B}$

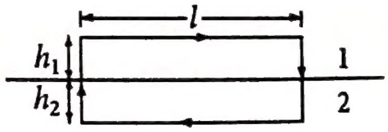

Fig. 17.4. Derivation of the boundary condition on the tangential component of $\vec{E}$

The boundary condition on the tangential component of the electric field can be derived from the Maxwell equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{17.68}$$

Consider a rectangular loop across the interface between the two media, as shown in Fig. 17.4. Integration of Eq. (17.68) over this loop gives

$$\int_S \vec{\nabla} \times \vec{E} \cdot \hat{n}\, dS = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, dS.$$

Applying Stokes's theorem to the left-hand side we obtain on making the lengths h_1 and h_2 very small

$$lE_{1t} - lE_{2t} = 0 \text{ or, } E_{1t} = E_{2t}, \tag{17.69}$$

i.e., *the tangential component of the electric field is continuous across the interface.*

The boundary condition for the normal component of the electric displacement can be obtained from the Maxwell equation

$$\vec{\nabla} \cdot \vec{D} = \rho. \tag{17.70}$$

Consider a pill-box-shaped surface, as shown in Fig. 17.3. Integrating Eq. (17.70) over the volume enclosed by the pill-box, applying the divergence theorem and making the height of the pill-box very small so that the side surface area of the pill-box is negligibly small, we obtain

$$D_{1n} - D_{2n} = \sigma, \tag{17.71}$$

where σ is the surface charge density on the interface. We have the continuity equation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \tag{17.72}$$

where $\vec{J}$ is the current density and ρ is the volume charge density. Integrating Eq. (17.72) over the volume enclosed by the pill-box and making the height of the pill-box negligible, we get, as in the case of Eq. (17.70),

$$J_{1n} - J_{2n} = -\frac{\partial \sigma}{\partial t}. \tag{17.73}$$

For monochromatic waves, σ will vary as $e^{-i\omega t}$, so that Eqs. (17.71) and (17.73) can be written as

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \sigma \tag{17.74}$$

$$\text{and } \sigma_{c1} E_{1n} - \sigma_{c2} E_{2n} = i\omega\sigma, \tag{17.75}$$

where the relationships $\vec{D} = \varepsilon \vec{E}$ and $\vec{J} = \sigma_c \vec{E}$ have been used.

We consider the following cases:

(i) If $\sigma = 0$, Eqs. (17.74) and (17.75) give

$$\frac{\varepsilon_1}{\sigma_{c1}} = \frac{\varepsilon_2}{\sigma_{c2}}, \tag{17.76}$$

which can be satisfied for properly chosen materials, or when $\sigma_{c1} = \sigma_{c2} = 0$ or ∞. While perfectly good conductors are not obtained in practice, good dielectrics are obtainable. Hence the case $\sigma_{c1} = \sigma_{c2} = 0$ is of practical interest when we have the interface between two good dielectrics.

(ii) If $\sigma \neq 0$, we have to eliminate σ between Eqs. (17.74) and (17.75). The result is

$$\left(\varepsilon_1 + i\frac{\sigma_{c1}}{\omega}\right) E_{1n} = \left(\varepsilon_2 + i\frac{\sigma_{c2}}{\omega}\right) E_{2n}. \tag{17.77}$$

(iii) If $\sigma_{c2} = \infty$, then $E_{2n} = 0$, since the electric field inside a perfect conductor must vanish. We have from Eq. (17.74), $E_{1n} = \sigma/\varepsilon_1$, or, $D_{1n} = \sigma$.

The boundary condition for the tangential components of $\vec{H}$ is derived from the Maxwell equation

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{17.78}$$

We integrate this equation over the area of a loop like that shown in Fig. 17.4, and make h_1 and h_2 very small, as before. We then obtain the boundary condition

$$H_{1t} - H_{2t} = j_{\perp}, \tag{17.79}$$

where $j_{\perp}$ is the component of the surface current density in the direction perpendicular to the plane of the loop. The surface current density is nonvanishing only when the electrical conductivity is infinity. Therefore, for finite conductivity, we have

$$H_{1t} = H_{2t}. \tag{17.80}$$

Thus, *when the medium conductivity is finite, the tangential component of magnetic intensity is continuous across the interface.* If the conductivity of medium 2 is infinite, then $\vec{E}_2$ is zero. Since $\vec{\nabla} \times \vec{E}_2 = -\mu_2 \partial \vec{H}_2/\partial t$, the vanishing of $\vec{E}_2$ implies that $\vec{H}_2$ does not vary with time. In other words, we may take $\vec{H}_2 = 0$. Hence the boundary condition (17.79) gives

$$H_{1t} = j_{\perp}. \tag{17.81}$$

That is, *when medium 2 has infinite conductivity, the tangential component of the magnetic intensity in medium 1 at the interface is equal to the normal component of the surface current density.*

The boundary conditions, derived above, are summarised in Table 17.1 for ready reference.

Table 17.1: Boundary conditions for E and H

	E_t	D_n	H_t	B_n
When $\sigma_{c1} = \sigma_{c2} = 0$	$E_{1t} = E_{2t}$	$D_{1n} = D_{2n}$	$H_{1t} = H_{2t}$	$B_{1n} = B_{2n}$
When $\sigma_{c2} = \infty$	$E_{2t} = E_{1t} = 0$	$D_{2n} = 0, D_{1n} = \sigma$	$H_{2t} = 0, H_{1t} = j_1$	$B_{1n} = B_{2n} = 0$
When σ_{c1} and σ_{c2} are finite	$E_{1t} = E_{2t}$	$\left(\varepsilon_1 + i\frac{\sigma_{c1}}{\omega}\right) E_{1n} = \left(\varepsilon_2 + i\frac{\sigma_{c1}}{\omega}\right) E_{2n}$	$H_{1t} = H_{2t}$	$B_{1n} = B_{2n}$

17.10 Plane Monochromatic Waves in a Conducting Medium

The wave equation for a conducting medium is given by Eq. (17.26). For monochromatic waves, we can write

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r})\, e^{-i\omega t}, \tag{17.82}$$

of course remembering that the physical electric field is given by the real part of Eq. (17.82). Substituting for $\vec{E}(\vec{r}, t)$ from Eq. (17.82) into Eq. (17.26) we obtain

$$\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} + i\omega\sigma_c \mu \vec{E} = 0, \tag{17.83}$$